

CURVATURE PROPERTIES OF WEYL GEOMETRIES

PETER GILKEY, STANA NIKČEVIĆ, AND UDO SIMON

ABSTRACT. We examine relations between geometry and the associated curvature decompositions in Weyl Geometry.

MSC 2002: 53B05, 15A72, 53A15, 53B10, 53C07, 53C25

**Dedicated to Heinrich Wefelscheid
on the occasion of his 70th birthday**

1. Introduction

1.1. Weyl geometry. Let N be a smooth manifold of dimension $n \geq 3$. Let ∇ be a torsion free connection on the tangent bundle TN of N and let g be a semi-Riemannian metric on N . Then the triple $\mathcal{W} := (N, g, \nabla)$ is said to be a *Weyl manifold* if there exists a smooth 1-form $\phi \in C^\infty(T^*N)$ so that:

$$\nabla g = -2\phi \otimes g. \quad (1.a)$$

Weyl geometry [12] is linked with conformal geometry. If $f \in C^\infty(N)$, let $g_1 := e^{2f}g$ be a conformally equivalent metric. If $\mathcal{W} = (N, g, \nabla)$ is a Weyl manifold, then the triple (N, g_1, ∇) is again a Weyl manifold where the associated 1-form is given by taking $\phi_1 := \phi - df$. The transformation of the pair $(g, \phi) \rightarrow (g_1, \phi_1)$ is called a *gauge transformation*. Properties of the Weyl geometry that are invariant under gauge transformations are called *gauge invariants*.

Let ∇^g be the Levi-Civita connection of g . There exists a conformally equivalent metric g_1 locally so that $\nabla = \nabla^{g_1}$ if and only if $d\phi = 0$; if $d\phi = 0$, such a class exists globally if and only if the associated de Rham cohomology class $[\phi]$ vanishes.

1.2. Affine and Riemannian geometry. We say that the pair $\mathcal{A} := (N, \nabla)$ is an *affine manifold* if ∇ is a torsion free connection on TN . Similarly, we say that the pair $\mathcal{N} := (N, g)$ is a *semi-Riemannian manifold* if g is a semi-Riemannian metric on N . Weyl geometry lies between affine geometry and semi-Riemannian geometry. Every Weyl manifold gives rise both to an underlying affine manifold (N, ∇) and to an underlying semi-Riemannian manifold (N, g) ; Equation (1.a) provides the link between these two structures. Since the Levi-Civita connection ∇^g is torsion free and since $\nabla^g g = 0$, the triple (N, g, ∇^g) is a Weyl manifold. There are, however, examples with $d\phi \neq 0$, so Weyl geometry is more general than semi-Riemannian geometry or even than conformal semi-Riemannian geometry.

1.3. Curvature. The *curvature operator* \mathcal{R} of a torsion free connection ∇ is the element of $\otimes^2 T^*N \otimes \text{End}(TN)$ which is defined by:

$$\mathcal{R}(x, y)z := (\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]})z.$$

The g -associated *curvature tensor* is given by using the metric to lower an index:

$$R(x, y, z, w) := g(\mathcal{R}(x, y)z, w).$$

We have the following identities:

$$R(x, y, z, w) + R(y, x, z, w) = 0, \quad \text{and} \quad (1.b)$$

$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \quad (1.c)$$

The relation of Equation (1.c) is called the *Bianchi identity*. The *Ricci tensor* $\text{Ric} := \text{Ric}(R) := \text{Ric}(\mathcal{R})$ is defined by setting:

$$\text{Ric}(x, y) := \text{Tr}\{z \rightarrow (\mathcal{R}(z, x)y)\}.$$

The tensor $\text{Ric}(\mathcal{R})$ does not depend on the metric g and is a gauge invariant. Let $g_{ij} := g(e_i, e_j)$ give the components of the metric tensor g relative to a local frame $\{e_i\}$ for TN . Let g^{ij} be the components of the inverse matrix g^{-1} relative to the dual frame $\{e^i\}$ on T^*N . We adopt the *Einstein convention* and sum over repeated indices. We then express:

$$\text{Ric}(x, y) = g^{ij} R(e_i, x, y, e_j).$$

We contract again to define the *scalar curvature* of R with respect to g by setting:

$$\tau_g := \text{Tr}_g \text{Ric} := g^{ij} \text{Ric}(e_i, e_j).$$

We can define another Ricci-type tensor $\text{Ric}^* := \text{Ric}^*(R)$ by setting:

$$\text{Ric}^*(x, y) := g^{ij} R(x, e_i, e_j, y).$$

We let R_{ij} and R_{ij}^* be the components of these two tensors:

$$R_{ij} := \text{Ric}(e_i, e_j) \quad \text{and} \quad R_{ij}^* := \text{Ric}^*(e_i, e_j).$$

We may decompose any (0,2)-tensor θ in the form $\theta = S\theta + \Lambda\theta$ where $S\theta$ and $\Lambda\theta$ are the symmetrization and the anti-symmetrization, respectively, of θ . We have the following result (see, for example, the discussion in [6, 12]):

Theorem 1.1. *Let (N, g, ∇) be a Weyl manifold. Let $R = R(\nabla)$. Then we have*

$$\begin{aligned} R(x, y, z, w) + R(x, y, w, z) &= \frac{2}{n} \{\text{Ric}(y, x) - \text{Ric}(x, y)\} g(z, w) \\ &= -\frac{4}{n} (\Lambda \text{Ric})(x, y) g(z, w), \end{aligned} \quad (1.d)$$

$$S \text{Ric}^* = S \text{Ric}, \quad \Lambda \text{Ric}^* = \frac{n-4}{n} \Lambda \text{Ric}, \quad \text{and} \quad \tau_g = \text{Tr}_g \text{Ric}^*. \quad (1.e)$$

Remark 1.2. If $\nabla = \nabla^g$ arises from semi-Riemannian geometry, then one has an additional symmetry:

$$R(x, y, z, w) + R(x, y, w, z) = 0. \quad (1.f)$$

1.4. The algebraic context. It is convenient to work in an abstract algebraic setting. Let V be a real vector space of dimension $n \geq 3$, with a non-degenerate scalar product of signature (p, q) :

$$h : V \times V \rightarrow \mathbb{R}.$$

Let $\mathfrak{R} = \mathfrak{R}(V) \subset \otimes^4 V^*$ be the space of all *generalized curvature tensors*. An element $A \in \otimes^4(V^*)$ belongs to \mathfrak{R} if and only if A satisfies the relations of Equation (1.b) and Equation (1.c). In what follows, we will use A and \mathcal{A} , respectively, when working in the abstract algebraic context, and we will use R and \mathcal{R} , respectively, when working in the geometric context; analogously we use h and g , respectively, to raise and lower indices as needed.

The subspace $\mathfrak{W} \subset \mathfrak{R}$ of *Weyl tensors* is defined by imposing, in addition to the relations of Equations (1.b) and (1.c), the symmetry of Equation (1.d). The subspace of *algebraic curvature tensors* $\mathfrak{A} \subset \mathfrak{R}$ is defined by imposing, additionally the relations of Equation (1.b) and of Equation (1.c), the symmetry of Equation (1.f); elements of \mathfrak{A} are said to be *algebraic*. Note that

$$\mathfrak{A} \subset \mathfrak{W} \subset \mathfrak{R}.$$

We shall see in Section 2 that these are proper containments if $n \geq 4$.

Let $A \in \mathfrak{R}(V)$. In the presence of Equations (1.b) and (1.c) the relations of Equation (1.f) and the curvature symmetry $A(x, y, z, w) = A(z, w, x, y)$ are equivalent, see, for example, the discussion in [5]. Consequently, it is useful to introduce the *conjugate curvature tensor* A^* by setting:

$$A^*(x, y, w, z) = -A(x, y, z, w).$$

Note that the conjugate curvature tensor does not necessarily satisfy the Bianchi identity given in Equation (1.c) (see, for example, the discussion in [5] Section 2.3). Consequently, A^* in general is not an element of $\mathfrak{R}(V)$. We raise indices to define \mathcal{A}^* ; \mathcal{A}^* is characterized by the identity:

$$h(\mathcal{A}^*(x, y)z, w) := -A(x, y, w, z).$$

We introduce a convenient notation from the physics literature and set

$$F := -\frac{2}{n} \Lambda \text{Ric}.$$

Let $h_{ij} := h(e_i, e_j)$. If $A \in \mathfrak{W}$, we have:

$$\begin{aligned} A_{ijkl}^* + A_{ijlk}^* &= -2F_{ij} h_{kl}, \\ A_{ijkl}^* + A_{jkil}^* + A_{kijl}^* &= \frac{4}{n} (\Lambda R_{ij} h_{kl} + \Lambda R_{jk} h_{il} + \Lambda R_{ki} h_{jl}) \\ &= -2(F_{ij} h_{kl} + F_{jk} h_{il} + F_{ki} h_{jl}). \end{aligned}$$

The following Proposition was proved in [6]:

Proposition 1.3. *Let $n \geq 3$. Let $A \in \mathfrak{W}$. The following assertions are equivalent:*

- (1) $A \in \mathfrak{A}$. (2) $A^* \in \mathfrak{A}$. (3) A^* satisfies the Bianchi identity (1.c).

The Proposition implies (see [6]):

Theorem 1.4. *Let $n \geq 3$. Let $\mathcal{W} = (N, g, \nabla)$ be a Weyl manifold. Assume that $H^1(N; \mathbb{R}) = 0$ and that the conjugate curvature tensor R^* is a generalized curvature tensor. Then there exists $f \in C^\infty(N)$ so that the Weyl connection ∇ is the Levi-Civita connection of the conformally equivalent metric $e^{2f}g$.*

The definition of the conjugate curvature tensor implies $\text{Ric}^*(R) = \text{Ric}(R^*)$. Moreover, Theorem 1.1 implies:

Proposition 1.5. *Let (N, g, ∇) be a Weyl manifold. Then:*

- (1) *If $n = 4$, then $\Lambda \text{Ric}^* = 0$ and thus Ric^* is symmetric.*
(2) *If $n \neq 4$, then R^* satisfies the relation:*

$$R_{ijlk}^* + R_{ijkl}^* = \frac{4}{n-4} \cdot \Lambda R_{ij}^* \cdot g_{kl}.$$

We say that the triple $\mathcal{W} := (V, h, A)$ is a *Weyl model* if $A \in \mathfrak{W}$. We say that such a triple is *geometrically realized* by the Weyl manifold $\mathcal{W} = (N, g, \nabla)$ if there exists a point $P \in N$ and an isomorphism $\Phi : V \rightarrow T_P N$ so that $\Phi^* g_P = h$ and so that $\Phi^* R_P = A$. One can pass from the algebraic setting to the geometric setting using the following result [6].

Theorem 1.6. *Every Weyl model is geometrically realized by a Weyl manifold.*

Here is a brief outline to the remainder of the paper. In Section 2, we derive the curvature decomposition of Higa [8] for Weyl manifolds; we shall discuss Higa's result in the context of the decomposition results from [1, 5]. In Section 3, we study further curvature properties and various special classes of *Weyl manifolds*. We shall apply Higa's gauge invariant *canonical metric* and prove several global results. We shall also study Einstein-Weyl manifolds and projectively flat Weyl connections. We conclude our discussion in Section 4 applying our decomposition results to the study of the well known gauge invariant curvatures, the *directional curvature* and the *length curvature*. Set $\mathcal{F}(x, y)z := F(x, y)z$.

2. Curvature decompositions

We recall some results from [2] related to earlier results of Singer and Thorpe [11] (see also the discussion in [5]). Let $\mathcal{O} := O(V, h)$ be the associated *orthogonal group*.

Definition 2.1. Set

- (1) $S^2 := \{\theta \in \otimes^2(V^*) : \theta_{ij} = \theta_{ji}\}.$
- (2) $S_0^2 := \{\theta \in \otimes^2(V^*) : \theta_{ij} = \theta_{ji}, \quad h^{ij}\theta_{ij} = 0\}.$
- (3) $\Lambda^2 := \{\theta \in \otimes^2(V^*) : \theta_{ij} = -\theta_{ji}\}.$
- (4) $W_6 := \{A \in \otimes^4(V^*) : A_{ijkl} = -A_{jikl} = A_{klij}, \quad h^{il}A_{ijkl} = 0, \\ A_{ijkl} + A_{jkil} + A_{kijl} = 0\}.$
- (5) $W_7 := \{A \in \otimes^4(V^*) : A_{ijkl} = -A_{jikl} = A_{ijlk}, \quad h^{il}A_{ijkl} = 0, \\ A_{kjil} + A_{ikjl} - A_{ljik} - A_{iljk} = 0\}.$
- (6) $W_8 := \{A \in \otimes^4(V^*) : A_{ijkl} = -A_{jikl} = -A_{klij}, \quad h^{il}A_{ijkl} = 0\}.$

Note that W_6 and W_7 are submodules of \mathfrak{R} whereas W_8 is not a submodule of \mathfrak{R} .

2.1. Decompositions of \mathfrak{A} and \mathfrak{R} .

Theorem 2.2. Let $n \geq 4$.

- (1) The modules $\{\mathbb{R}, S_0^2, \Lambda^2, W_6, W_7, W_8\}$ are inequivalent and irreducible \mathcal{O} modules.
- (2) There is an \mathcal{O} module isomorphism $\mathfrak{R} \approx \mathbb{R} \oplus 2 \cdot S_0^2 \oplus 2 \cdot \Lambda^2 \oplus W_6 \oplus W_7 \oplus W_8$.
- (3) There is an \mathcal{O} module isomorphism $\mathfrak{A} \approx \mathbb{R} \oplus S_0^2 \oplus W_6$.

Remark 2.3. If $n = 3$, we set $W_6 = W_8 = 0$ to obtain the corresponding decomposition. For $n \geq 5$, the modules of Assertion (1) in Theorem 2.2 are also irreducible $SO(V, h)$ modules; if $n = 4$, we must decompose $W_6 = W_6^+ \oplus W_6^-$ as the sum of the dual and anti-self dual Weyl conformal curvature tensors. The space W_6 is the space of all *Weyl conformal curvature tensors*. One has:

$$\begin{aligned} \dim\{\mathfrak{R}\} &= \frac{1}{3}n^2(n^2 - 1), & \dim\{\Lambda^2\} &= \frac{n(n-1)}{2}, \\ \dim\{\mathfrak{A}\} &= \frac{1}{12}n^2(n^2 - 1), & \dim\{W_6\} &= \frac{n(n+1)(n-3)(n+2)}{12}, \\ \dim\{\mathbb{R}\} &= 1, & \dim\{W_7\} &= \frac{(n-1)(n-2)(n+1)(n+4)}{8}, \\ \dim\{S_0^2\} &= \frac{(n-1)(n+2)}{2}, & \dim\{W_8\} &= \frac{n(n-1)(n-3)(n+2)}{8}. \end{aligned}$$

Proof. We sketch the proof of Theorem 2.2. Define $\pi_{\Lambda \otimes S} : \mathfrak{R}(V) \rightarrow \Lambda^2 \otimes S^2$ by:

$$\pi_{\Lambda \otimes S}(A)(x, y, z, w) := \frac{1}{2}\{A(x, y, z, w) + A(x, y, w, z)\}.$$

One verifies easily that $\ker(\pi_{\Lambda \otimes S}) = \mathfrak{A}(V)$. Let $\Theta \in \Lambda^2 \otimes S^2$. Define

$$\{\sigma_{\Lambda \otimes S}(\Theta)\}_{ijkl} := \Theta_{ijkl} + \frac{1}{2}\{\Theta_{kjil} + \Theta_{ikjl} - \Theta_{ljik} - \Theta_{iljk}\}.$$

Let $A = \sigma_{\Lambda \otimes S}\Theta$. Clearly $A_{ijkl} = -A_{jikl}$. We verify the Bianchi identity is satisfied and show thereby that $\sigma_{\Lambda \otimes S} : \Lambda^2 \otimes S^2 \rightarrow \mathfrak{R}$ by computing:

$$\begin{aligned} A_{ijkl} + A_{jkil} + A_{kijl} &= \Theta_{ijkl} + \frac{1}{2}\{\Theta_{kjil} + \Theta_{ikjl} - \Theta_{ljik} - \Theta_{iljk}\} \\ &\quad + \Theta_{jkil} + \frac{1}{2}\{\Theta_{ikjl} + \Theta_{jikl} - \Theta_{lkji} - \Theta_{jlki}\} \\ &\quad + \Theta_{kijl} + \frac{1}{2}\{\Theta_{jikl} + \Theta_{kjil} - \Theta_{likj} - \Theta_{klij}\} \\ &= \Theta_{ijkl}(1 - \frac{1}{2} - \frac{1}{2}) + \Theta_{jkil}(1 - \frac{1}{2} - \frac{1}{2}) + \Theta_{kijl}(1 - \frac{1}{2} - \frac{1}{2}) \\ &\quad + \Theta_{ljik}(-\frac{1}{2} + \frac{1}{2}) + \Theta_{iljk}(-\frac{1}{2} + \frac{1}{2}) + \Theta_{klji}(-\frac{1}{2} + \frac{1}{2}) = 0. \end{aligned}$$

We show that $\sigma_{\Lambda \otimes S}$ is a splitting of $\pi_{\Lambda \otimes S}$ by checking:

$$\begin{aligned}
& (\pi_{\Lambda \otimes S} \sigma_{\Lambda \otimes S} \Theta)_{ijkl} \\
&= \frac{1}{2} \Theta_{ijkl} + \frac{1}{4} \{ \Theta_{kjil} + \Theta_{ikjl} - \Theta_{ljik} - \Theta_{iljk} \} \\
&+ \frac{1}{2} \Theta_{ijlk} + \frac{1}{4} \{ \Theta_{ljik} + \Theta_{iljk} - \Theta_{kjil} - \Theta_{ikjl} \} = \Theta_{ijkl}.
\end{aligned}$$

Consequently, $\pi_{\Lambda \otimes S}$ is a surjective map and thus we have an \mathcal{O} module isomorphism:

$$\mathfrak{R} \approx \mathfrak{A} \oplus (\Lambda^2 \otimes S^2).$$

The Theorem then follows from the decomposition of \mathfrak{A} as an \mathcal{O} module [11] and by decomposing $\Lambda^2 \otimes S^2$ as an \mathcal{O} module; we omit details in the interests of brevity. \square

Definition 2.4. If $\varphi \in \Lambda^2$, set

- (1) $\sigma_4(\varphi)(x, y, z, w) := 2\varphi(x, y)h(z, w) + \varphi(x, z)h(y, w) - \varphi(y, z)h(x, w).$
- (2) $\sigma_5(\varphi)(x, y, z, w) := \varphi(x, w)h(y, z) - \varphi(y, w)h(x, z).$

Lemma 2.5. If $\varphi \in \Lambda^2$ then $\sigma_4(\varphi) \in \mathfrak{R}$ and $\sigma_5(\varphi) \in \mathfrak{R}$.

Proof. Let $\varphi \in \Lambda^2$. Let $A_4 := \sigma_4(\varphi)$ and $A_5 := \sigma_5(\varphi)$. It is then immediate that $A_i(x, y, z, w) = -A_i(y, x, z, w)$. We establish the Bianchi identity by computing:

$$\begin{aligned}
& A_4(x, y, z, w) + A_4(y, z, x, w) + A_4(z, x, y, w) \\
&= 2\varphi(x, y)h(z, w) + \varphi(x, z)h(y, w) - \varphi(y, z)h(x, w) \\
&+ 2\varphi(y, z)h(x, w) + \varphi(y, x)h(z, w) - \varphi(z, x)h(y, w) \\
&+ 2\varphi(z, x)h(y, w) + \varphi(z, y)h(x, w) - \varphi(x, y)h(z, w) = 0, \\
& A_5(x, y, z, w) + A_5(y, z, x, w) + A_5(z, x, y, w) \\
&= \varphi(x, w)h(y, z) - \varphi(y, w)h(x, z) \\
&+ \varphi(y, w)h(z, x) - \varphi(z, w)h(y, x) \\
&+ \varphi(z, w)h(x, y) - \varphi(x, w)h(z, y) = 0.
\end{aligned}$$

\square

2.2. Higa's decomposition of \mathfrak{W} . We can now discuss the decomposition of \mathfrak{W} as an \mathcal{O} module. We reformulate a result of Higa [8].

Theorem 2.6. For $n \geq 4$, we may decompose $\mathfrak{W} = \mathfrak{A} \oplus \mathfrak{P}$ where

$$\mathfrak{P} = \{(\sigma_4 - \sigma_5)\varphi\}_{\varphi \in \Lambda^2}.$$

Consequently, $\mathfrak{W} \approx \mathbb{R} \oplus S_0^2 \oplus \Lambda^2 \oplus W_6$ as an \mathcal{O} module.

Proof. In the proof of Theorem 2.2, we constructed a short exact sequence

$$0 \rightarrow \mathfrak{A}(V) \rightarrow \mathfrak{R}(V) \xrightarrow{\pi_{\Lambda \otimes S}} \Lambda^2 \otimes S^2 \rightarrow 0.$$

Equation (1.d) then implies that $\pi_{\Lambda \otimes S}(A)$ takes values in $\Lambda^2 \otimes \mathbb{R} \cdot h$. It is immediate from the definition that $\mathfrak{A} \subset \mathfrak{W}$, and thus either $\mathfrak{W} = \mathfrak{A}$ or \mathfrak{W} is isomorphic to $\mathfrak{A} \oplus \Lambda^2$ as an \mathcal{O} module. We argue that this latter possibility pertains. If $\varphi \in \Lambda^2(V)$, define:

$$\begin{aligned}
A_{ijkl} &:= (\sigma_4\varphi - \sigma_5\varphi)_{ijkl} \\
&= 2\varphi_{ij}h_{kl} + \varphi_{ik}h_{jl} - \varphi_{jk}h_{il} - \varphi_{il}h_{jk} + \varphi_{jl}h_{ik}.
\end{aligned}$$

We define the Ricci type components $A_{jk} := \text{Ric}(A)_{jk} := h^{il}A_{ijkl}$ and compute:

$$\begin{aligned}
A_{jk} &= h^{il}\{2\varphi_{ij}h_{kl} + \varphi_{ik}h_{jl} - \varphi_{jk}h_{il} - \varphi_{il}h_{jk} + \varphi_{jl}h_{ik}\} \\
&= 2\varphi_{kj} + \varphi_{jk} - n\varphi_{jk} + \varphi_{jk} = -n\varphi_{jk}, \\
A_{ijkl} + A_{ijlk} &= 4\varphi_{ij}h_{kl} = \frac{2}{n}(A_{ji} - A_{ij})h_{kl}.
\end{aligned}$$

\square

Consider Higa's decomposition $\mathfrak{W} = \mathfrak{A} \oplus \mathfrak{P}$ from above and the orthogonal projections $\pi_{\mathfrak{A}} : \mathfrak{W} \rightarrow \mathfrak{A}$ and $\pi_{\mathfrak{P}} : \mathfrak{W} \rightarrow \mathfrak{P}$. For $A \in \mathfrak{W}$ we call $H(A) := \pi_{\mathfrak{P}}(A)$ the *Higa term* of A . In Section 2.7 we will relate $H(A)$ with the decompositions that we study in the next Sections 2.3 - 2.5.

2.3. The A -decomposition and the W -decomposition for \mathfrak{W} . In Theorem 2.2, we identified the decomposition factors of \mathfrak{R} as an \mathcal{O} module. However, since the modules S_0^2 and Λ^2 both appear with multiplicity 2, the decomposition of \mathfrak{R} is not unique. In [5] we studied decompositions of \mathfrak{R} in some detail and, following the discussion in [2], presented two different possibilities for the decomposition of \mathfrak{R} :

$$\bigoplus_{i=1}^8 A_i = \mathfrak{R} = \bigoplus_{i=1}^8 W_i.$$

We denote the corresponding orthogonal projections by

$$\alpha_i : \mathfrak{R} \rightarrow A_i \quad \text{and} \quad \pi_i : \mathfrak{R} \rightarrow W_i.$$

Let $p(A) := \mathfrak{R} \cap \ker(\text{Ric})$ be the space of *projective curvature tensors*. While the A -decomposition also gives rise to a decomposition of \mathfrak{A} , it does not induce a decomposition of $p(A)$. On the other hand the W -decomposition induces a decomposition of $p(A)$ but does not induce a decomposition of \mathfrak{A} . Thus both decompositions are important in the geometric study of manifolds and their curvature properties. Again we emphasize that this is possible because both S_0^2 and Λ^2 appear with multiplicity 2 in the decomposition of \mathfrak{R} ; thus identifying the exact subspace of \mathfrak{R} which is isomorphic to S_0^2 or to Λ^2 in the submodules \mathfrak{A} , $p(A)$, and \mathfrak{W} is crucial. We follow the discussion in [5] of these two inequivalent decompositions and evaluate them for Weyl manifolds. We begin by establishing some useful notational conventions:

Definition 2.7. Let θ_1 and θ_2 be bilinear forms.

- (1) Define $(\theta_1 \cdot \theta_2)(x, y, z, w) := \theta_1(x, y)\theta_2(z, w)$.
- (2) Define $(\theta_1 \wedge_r \theta_2)(x, y, z, w) := \theta_1(x, z)\theta_2(y, w) - \theta_1(y, z)\theta_2(x, w) - r[\theta_1(x, w)\theta_2(y, z) - \theta_1(y, w)\theta_2(x, z)]$ for $r \in \mathbb{N}$, and $\wedge := \wedge_0$.
- (3) Define mappings ψ and μ from $\otimes^4 V^*$ to $\otimes^4 V^*$ by setting

$$\begin{aligned} 4\psi(A)(x, y, z, w) &:= A(x, y, z, w) + A(y, x, w, z) \\ &\quad + A(z, w, x, y) + A(w, z, y, x), \quad \text{and} \\ 8\mu(A)(x, y, z, w) &:= 3A(x, y, z, w) + 3A(x, y, w, z) \\ &\quad + A(x, w, z, y) + A(x, z, w, y) + A(w, y, z, x) + A(z, y, w, x). \end{aligned}$$

Remark 2.8.

- (1) Note $\theta_1 \wedge_k \theta_1 = (k+1)\theta_1 \wedge \theta_1$.
- (2) $\theta_1 \wedge_r \theta_2 + \theta_1 \wedge_s \theta_2 = \theta_1 \wedge \theta_2 + \theta_1 \wedge_{r+s} \theta_2$.

2.4. The A -decomposition for \mathfrak{W} . Recall that α_i is orthogonal projection on the component A_i .

Lemma 2.9. *If $A \in \mathfrak{W}$, then:*

- (1) $\alpha_1(A) = -\frac{1}{n(n-1)}\tau_h h \wedge h$.
- (2) $\alpha_2(A) = \frac{-1}{2(n-2)}S(\text{Ric} + \text{Ric}^*) \wedge_1 h + \frac{2}{n(n-2)}\tau_h h \wedge h$
 $= -\frac{1}{n-2}(S \text{Ric} \wedge_1 h) + \frac{2}{n(n-2)}\tau_h h \wedge h$.
- (3) $\alpha_3(A) = -\frac{1}{2n}S(\text{Ric} - \text{Ric}^*) \wedge_{-1} h = 0$.
- (4) $\alpha_4(A) = -\frac{1}{4(n+2)}[2\Lambda(3 \text{Ric} - \text{Ric}^*) \cdot h + \Lambda(3 \text{Ric} - \text{Ric}^*) \wedge_{-1} h]$
 $= -\frac{1}{2n}(2(\Lambda \text{Ric}) \cdot h + (\Lambda \text{Ric}) \wedge_{-1} h)$.
- (5) $\alpha_5(A) = -\frac{1}{4(n-2)}[2\Lambda(\text{Ric} + \text{Ric}^*) \cdot h + \Lambda(\text{Ric} + \text{Ric}^*) \wedge_3 h]$
 $= -\frac{1}{2n}(2(\Lambda \text{Ric}) \cdot h + (\Lambda \text{Ric}) \wedge_3 h)$.

- (6) $\alpha_6(A) = \psi(A) - \alpha_1(A) - \alpha_2(A)$
 $= A + \frac{2}{n} (\Lambda \text{Ric}) \cdot h + \frac{1}{n} (\Lambda \text{Ric}) \wedge_1 h + \frac{1}{n-2} (S \text{Ric}) \wedge_1 h - \frac{1}{(n-1)(n-2)} \tau_h h \wedge h.$
- (7) $\alpha_7(A) = 0.$
- (8) $\alpha_8(A) = 0.$

Remark 2.10. Let $A \in \mathfrak{W}$. In analogy to the Schouten tensor in conformal semi-Riemannian geometry introduce the symmetric *Weyl-Schouten tensor* by setting:

$$\sigma := \frac{1}{n-2} \left[S \text{Ric} - \frac{1}{2(n-1)} \cdot \tau_h h \right].$$

We then have $\alpha_6(A) = A + \sigma \wedge_1 h - (\alpha_4(A) + \alpha_5(A))$. This decomposition extends the well known decomposition of the Weyl conformal curvature tensor in semi-Riemannian geometry; note that $\alpha_6(A)$ and also $\sigma \wedge_1 h$ are algebraic curvature tensors. We may express $\alpha_2(A) = -\sigma \wedge_1 h + \frac{1}{n(n-1)} \tau_h h \wedge h$.

Lemma 2.9 implies:

Lemma 2.11. *If $A \in \mathfrak{W}$ then:*

- (1) $\text{Ric}(\alpha_1(A)) = \text{Ric}^*(\alpha_1(A)) = \frac{1}{n} \tau_h h.$
- (2) $\text{Ric}(\alpha_2(A)) = \text{Ric}^*(\alpha_2(A)) = -\frac{1}{n} \tau_h h + S(\text{Ric}).$
- (3) $\text{Ric}(\alpha_4(A)) = -\text{Ric}^*(\alpha_4(A)) = \frac{n+2}{2n} \Lambda(\text{Ric}).$
- (4) $\text{Ric}(\alpha_5(A)) = \frac{1}{3} \text{Ric}^*(\alpha_5(A)) = \frac{n-2}{2n} \Lambda(\text{Ric}).$
- (5) $\text{Ric}(\alpha_j(A)) = \text{Ric}^*(\alpha_j(A)) = 0$ for $j = 3, 6, 7, 8.$

2.5. The W -decomposition for \mathfrak{W} . Recall that π_i is orthogonal projection on the component W_i .

Lemma 2.12. *If $A \in \mathfrak{W}$ then:*

- (1) $\pi_1(A) = \frac{1}{n(n-1)} \tau_h h \wedge h.$
- (2) $\pi_2(A) = \frac{1}{n-1} \left[\frac{1}{n} \tau_h h - S \text{Ric} \right] \wedge h.$
- (3) $\pi_3(A) = -\frac{1}{n+1} [2\Lambda \text{Ric} \cdot h + \Lambda \text{Ric} \wedge h].$
- (4) $\pi_4(A) = -\frac{1}{n^2-4} [2\Lambda \text{Ric}^* \cdot h + \Lambda \text{Ric}^* \wedge_{n+1} h]$
 $-\frac{3}{(n^2-4)(n+1)} [2\Lambda \text{Ric} \cdot h + \Lambda \text{Ric} \wedge_{n+1} h]$
 $= -\frac{1}{n(n+1)} [2(\Lambda \text{Ric}) \cdot h + \Lambda \text{Ric} \wedge_{n+1} h].$
- (5) $\pi_5(A) = \frac{1}{(n-1)(n-2)} [\tau_h h \wedge h - \frac{1}{n} S(\text{Ric} + (n-1) \text{Ric}^*) \wedge_{n-1} h]$
 $= \frac{1}{(n-1)(n-2)} [\tau_h h \wedge h - S \text{Ric} \wedge_{n-1} h].$
- (6) $\pi_6(A) = \psi(A) + \frac{1}{2(n-2)} S(\text{Ric} + \text{Ric}^*) \wedge_1 h - \frac{1}{(n-1)(n-2)} \tau_h h \wedge h$
 $= A + \frac{2}{n} (\Lambda \text{Ric}) \cdot h + \frac{1}{n} (\Lambda \text{Ric}) \wedge_1 h + \frac{1}{n-2} S \text{Ric} \wedge_1 h - \frac{1}{(n-1)(n-2)} \tau_h h \wedge h.$
- (7) $\pi_7(A) = \mu(R) + \frac{1}{2n} S(\text{Ric} - \text{Ric}^*) \wedge_{-1} h + \frac{1}{2(n+2)} \Lambda(3 \text{Ric} - \text{Ric}^*) \cdot h$
 $+ \frac{1}{4(n+2)} \Lambda(3 \text{Ric} - \text{Ric}^*) \wedge_{-1} h = 0.$
- (8) $\pi_8(A) = A - \psi(A) - \mu(A) + \frac{1}{2(n-2)} \Lambda(\text{Ric} + \text{Ric}^*) \cdot h$
 $+ \frac{1}{4(n-2)} \Lambda(\text{Ric} + \text{Ric}^*) \wedge_3 h = 0.$

Remark 2.13. If $A \in \mathfrak{W}$ satisfies $\text{Ric}(A) = 0$ then $\pi_i(A) = 0$ for $i \neq 6$ and $\pi_6(A) = A$.

Lemma 2.14. *If $A \in \mathfrak{W}$ then:*

- (1) $\text{Ric}(\pi_1(A)) = \text{Ric}^*(\pi_1(A)) = \frac{1}{n}\tau_h h$.
- (2) $\text{Ric}(\pi_2(A)) = S \text{Ric} - \frac{1}{n}\tau_h h = -(n-1) \text{Ric}^*(\pi_2(A))$.
- (3) $\text{Ric}(\pi_3(A)) = \Lambda \text{Ric} = -\frac{n+1}{3} \text{Ric}^*(\pi_3(A))$.
- (4) $\text{Ric}(\pi_4(A)) = 0$.
- (5) $\text{Ric}^*(\pi_4(A)) = \frac{(n-2)(n+2)}{n(n+1)} \Lambda \text{Ric}$.
- (6) $\text{Ric}(\pi_5(A)) = 0$.
- (7) $\text{Ric}^*(\pi_5(A)) = \frac{1}{n-1}(nS \text{Ric} - \tau_h h) = \frac{n}{n-1} \text{Ric}(\pi_2(A)) = -n \text{Ric}^*(\pi_2(A))$.
- (8) $\text{Ric}(\pi_6(A)) = 0$.
- (9) $\text{Ric}^*(\pi_6(A)) = 0$.

Remark 2.15. The Ricci and Ricci* tensors of $\pi_i(A)$ and $\alpha_i(A)$ for $i = 1, \dots, 6$ have essentially only 3 non-trivial (i.e. non-vanishing) types:

- (1) Constant multiples of $\Lambda(\text{Ric}(A))$.
- (2) Constant multiples of $\tau_h h$.
- (3) Constant multiples of $(S \text{Ric}(A) - \frac{1}{n}\tau_h h)$.

2.6. The projective curvature tensor. From [2] and [5] one knows that the projective curvature tensor $p(A)$ of $A \in \mathfrak{A}$ can be recovered from the W -decomposition as follows:

$$p(A) = \bigoplus_{i=4}^8 \pi_i(A). \quad (2.a)$$

Let $A \in \mathfrak{W}$. Equation (2.a) and the results for π_i given above then yield easily the following result:

Lemma 2.16. *Let $A \in \mathfrak{W}$; then:*

- (1) $p(A) = \bigoplus_{i=4}^6 \pi_i(A)$.
- (2) $p(A) = A + \frac{1}{n+1} [2\Lambda \text{Ric} \cdot h + \Lambda \text{Ric} \wedge h] + \frac{1}{n-1} (S \text{Ric}) \wedge h$.

Proposition 2.17. *Let $A \in \mathfrak{W}$. If $p(A) = 0$, then $A = -\frac{1}{n(n-1)}\tau_h h \wedge h$.*

Proof. $p(A) = 0$ implies $\pi_i(A) = 0$ for $i = 4, 5, 6$; but $\pi_4(A) = 0$ gives $\Lambda \text{Ric} = 0$, and $\pi_5(A) = 0$ gives $S \text{Ric} = \frac{1}{n}\tau_h h$; this proves the Proposition. \square

2.7. The Higa term and the conjugate curvature tensor. We now relate the conjugate curvature tensor R^* and the *Higa term* $H(A)$ of A . As the algebraic part of $A \in \mathfrak{W}$ is given by $(\alpha_1 + \alpha_2 + \alpha_6)(A)$, Lemma 2.9 gives:

Lemma 2.18.

- (1) $H = H(A) = (\alpha_4 + \alpha_5)(A) \in \mathfrak{P}$.
- (2) $A = (\alpha_1 + \alpha_2 + \alpha_6)(A) + H(A)$.

Moreover, we set

$$D(x, y, z, w) := -(A(x, y, z, w) + A(x, y, w, z)).$$

Then the conjugate curvature tensor satisfies

$$A^*(x, y, z, w) = A(x, y, z, w) + D(x, y, z, w).$$

We use Equation (1.d) to express:

$$-D(x, y, z, w) = A(x, y, z, w) + A(x, y, w, z) = -\frac{4}{n} (\Lambda \text{Ric}(A))(x, y) h(z, w)$$

and note the symmetries:

$$D(x, y, z, w) = D(x, y, w, z) = -D(y, x, z, w).$$

Lemma 2.19. *Adopt the notation established above. Then:*

- (1) $-4H(x, y, z, w) = 2D(x, y, z, w) + D(x, z, y, w)$
 $-D(y, z, x, w) - D(x, w, y, z) + D(y, w, x, z)$
 $= 2D(x, y, z, w) - [D(x, y, z, w) + D(y, z, x, w) + D(z, x, y, w)]$
 $- \{D(x, w, y, z) + D(w, y, x, z) + D(y, x, w, z)\}.$
- (2) $D(x, y, z, w) = H(x, y, z, w) + H(x, y, w, z).$

In particular, (1) in the Lemma implies that D determines the Higa term H , while (2) implies that H determines D . In (1), note that the brackets $[\dots]$ are cyclic in (x, y, z) and $\{\dots\}$ are cyclic in (x, w, y) . This immediately gives the proof of (2). Finally, the definition of A^* , the Higa decomposition of A , and the symmetries of the algebraic part of A yield:

Lemma 2.20.

- (1) $A^*(x, y, z, w) + A^*(x, y, w, z) = D(x, y, w, z).$
- (2) $A^*(x, y, z, w) = (\alpha_1 + \alpha_2 + \alpha_6)(A)(x, y, z, w) - (\alpha_4 + \alpha_5)(A)(x, y, w, z)$
 $= (\alpha_1 + \alpha_2 + \alpha_6)(A)(x, y, z, w) - H(A)(x, y, w, z).$

Proposition 2.21. *Let $A \in \mathfrak{W}$, then the Higa term satisfies:*

$$\begin{aligned} H(A) &= \alpha_4(A) + \alpha_5(A) = \pi_3(A) + \pi_4(A) \\ &= -\frac{1}{n} (2(\Lambda \text{ Ric}) \cdot h + (\Lambda \text{ Ric}) \wedge_1 h) . \end{aligned}$$

Lemma 2.22. *Let $A \in \mathfrak{W}$. For $n \neq 2$ we have the following equivalences:*

- (1) $\Lambda \text{ Ric} = 0.$ (2) $\alpha_4(A) = 0.$ (3) $\alpha_5(A) = 0.$ (4) $H(A) = 0.$
- (5) $\pi_3(A) = 0.$ (6) $\pi_4(A) = 0.$ (7) $A \in \mathfrak{A}.$

3. Weyl Manifolds

3.1. Equivalent notions for Weyl manifolds. If ϕ is a smooth 1-form on a semi-Riemannian manifold (N, g) , the dual vector field ϕ^\sharp is characterized by the identity $g(x, \phi^\sharp) = \phi(x)$ for all tangent fields x . The following result (see, for example, Theorem 6 [6]) can be used to construct Weyl manifolds:

Theorem 3.1. *Let ∇ be a torsion free connection on a semi-Riemannian manifold (N, g) . Let ϕ be a smooth 1-form on N . The following assertions are equivalent, and if either is satisfied then (N, g, ∇) is a Weyl manifold:*

- (1) $\nabla g = -2\phi \otimes g.$
- (2) $\nabla_x y = \nabla_x^g y + \phi(x)y + \phi(y)x - g(x, y)\phi^\sharp.$

3.2. Weyl manifolds and curvature decompositions. We pass from the algebraic setting to the geometric setting; for each $P \in N$ the semi-Riemannian metric induces a scalar product on the tangent space, we simply identify $(T_P N, g) =: (V, h)$. A conformal change of the metric does not change the associated orthogonal group; $O(V, g) = O(V, c \cdot g)$ for any real $c \neq 0$. Moreover, for any metric g within the conformal class and at any $P \in N$ the decompositions of the Weyl curvature tensor R are bijectively associated to corresponding decompositions of the Weyl curvature operator \mathcal{R} , and these decompositions obviously are gauge invariant. We speak about the A -decomposition and the W -decomposition of \mathcal{R} , respectively. Furthermore, since the Ricci tensor is a GL invariant, it is a gauge invariant. These observations and our previous discussions immediately give:

Theorem 3.2. *Let $\mathcal{N} = (N, g, \nabla)$ be a Weyl manifold and let $\mathcal{R} = \mathcal{R}(\nabla)$. Then:*

- (1) *The A -decomposition of \mathcal{R} is gauge invariant. \mathcal{N} is Ricci flat if and only if $R = \alpha_6(R)$ for some (and hence any) g within the conformal class.*
- (2) *The W -decomposition of \mathcal{R} is gauge invariant. \mathcal{N} is Ricci flat if and only if $R = \pi_6(R)$ for some (and hence any) g within the conformal class.*

3.3. The second Bianchi identity. We recall the following well known result from [4], p. 56: *The curvature tensor of any torsion free connection satisfies the second Bianchi identity.* However, since $\nabla_k g^{rs} = 2\phi_k g^{rs}$, raising and lowering indices need not commute with ∇ -covariant differentiation. One therefore has:

Lemma 3.3. *Let $\mathcal{N} = (N, g, \nabla)$ be a Weyl manifold and let $R = R(\nabla)$. Then:*

- (1) $\nabla_m \mathcal{R}_{ijk}{}^l + \nabla_i \mathcal{R}_{jmk}{}^l + \nabla_j \mathcal{R}_{mik}{}^l = 0.$
- (2) $\nabla_m R_{ijkl} + \nabla_i R_{jmk}{}^l + \nabla_j R_{mik}{}^l = -2(\phi_m R_{ijkl} + \phi_i R_{jmk}{}^l + \phi_j R_{mik}{}^l).$

The following result is a consequence of Theorem 1.1 and of Lemma 3.3:

Corollary 3.4. *We have the relations:*

$$\begin{aligned} \nabla_m R_{jk} - \nabla_j R_{mk} + \nabla_i R_{jmk}{}^i &= 0, \\ \nabla_m (\Lambda R_{jk}) + \nabla_j (\Lambda R_{km}) + \nabla_k (\Lambda R_{mj}) &= 0. \end{aligned}$$

3.4. The canonical Weyl metric. Let $\mathcal{W} = (N, g, \phi)$ be a Weyl manifold. If $g_1 := e^{2f}g$ is a conformally equivalent metric, $\tau_{g_1} := e^{-2f}\tau_g$. Thus there is a gauge invariant, disjoint decomposition of N into three subsets $N = N_0 \cup N^+ \cup N^-$ where

$$N_0 := \{p \in N : \tau_g(p) = 0\} \quad \text{and} \quad N^\pm := \{p \in N : \pm\tau_g(p) > 0\}.$$

In the following we consider Weyl manifolds with $\tau_g \neq 0$ and restrict to the case $\tau_g > 0$, thus $N = N^+$; the case $\tau_g < 0$ can be handled analogously. We recall a definition of Higa [8]; his definition and our definition differ by a constant positive factor; Higa calls his metric the *canonical metric*; we use the following terminology:

Definition 3.5.

- (1) Let $\mathcal{W} = (N, g, \nabla)$ be a Weyl manifold with $\tau_g > 0$. We call the *gauge invariant metric* $\tilde{g} := \tau_g \cdot g$ the *canonical Weyl metric* of \mathcal{W} . The metric \tilde{g} is the unique element in the conformal class defined by g so that $\tau_{\tilde{g}} = 1$.
- (2) Let (g, ϕ) be a pair that generates \mathcal{W} with $\tau_g > 0$. We call $\tilde{\phi} := \phi - \frac{1}{2}d \ln \tau_g$ the gauge invariant Weyl 1-form of \mathcal{W} ; $\nabla \tilde{g} = -2\tilde{\phi} \otimes \tilde{g}$.

Remark 3.6. The definition of the canonical Weyl metric is gauge invariant (thus \tilde{g} is a distinguished metric in the conformal class) and therefore all invariants of its induced semi-Riemannian geometry are gauge invariant; the analogue is true for invariants constructed from the pair $(\tilde{g}, \tilde{\phi})$.

3.5. Curvature invariants of the Weyl metric. Throughout this section, we assume $\tau_g > 0$ and let $\tilde{g} := \tau_g \cdot g$ be the Weyl metric. We restrict to the Riemannian setting i.e. g is positive definite. We introduce the following notational conventions:

Definition 3.7. *Let g be a Riemannian metric. Let $\Xi_g := \det(g_{ij})^{1/2}$.*

- (1) *Let $\Delta_g := \Xi_g^{-1} \partial_{x_i} \Xi_g g^{ij} \partial_{x_j}$ be the Laplace-Beltrami operator.*
- (2) *Let $\kappa_g := \frac{1}{n(n-1)} \text{Tr}_g \text{Ric}_g$ be the normalized scalar curvature of g .*
- (3) *Let $\omega_g = \Xi_g dx^1 \dots dx^n$ be the Riemannian measure.*

The following result gives the *conformal scalar curvature relations*:

Proposition 3.8.

- (1) $n\kappa_{\tilde{g}} = n\tau_g^{-1}\kappa_g - \tau_g^{-2}\Delta_g \tau_g - \frac{1}{4}\tau_g^{-3}(n-6)\|\text{grad}_g \tau_g\|_g^2.$
- (2) $1 = \tau_{\tilde{g}} = n(n-1)\kappa_{\tilde{g}} - 2(n-1)\nabla_k^{\tilde{g}} \tilde{\phi}^k - (n-1)(n-2)\|\tilde{\phi}\|_{\tilde{g}}^2.$

Proof. These formulas can, of course, be derived from classical formulas in the literature. It is instructive, however, to give a direct derivation. We argue as follows to prove Assertion (1). Let g be a Riemannian metric. Let ϑ be a positive smooth function on M and let $g_1 := \vartheta g$. We must express κ_{g_1} in terms of κ_g .

Choose local coordinates so the Christoffel symbols of ∇^g vanish at the point P in question. A bit of thought then expresses $R^{g_1} = R^g + \mathcal{E}_1 + \mathcal{E}_2$ at P where \mathcal{E}_1 is quadratic in the first derivatives of ϑ and \mathcal{E}_2 is linear in the second derivatives of ϑ at P , respectively. This leads to a formula of the form:

$$\kappa_{g_1} = \vartheta^{-1} \kappa_g + a_1 \vartheta^{a_2} \Delta_g \vartheta + a_3 \vartheta^{a_4} \|\text{grad}_g \vartheta\|_g^2 \quad (3.a)$$

where $a_i = a_i(n)$ are certain universal constants which depend on the dimension, but not on the metric chosen, and which need to be determined; Assertion (1) will then follow by specializing to the case $\vartheta = \tau_g$. To determine these constants in the general setting, we may take $g = dx_1^2 + \dots + dx_n^2$ to be flat. We take as a special case $\vartheta = \vartheta(x_1)$ and determine the coefficients in Equation (3.a) by computing:

$$\begin{aligned} \Gamma_{111} &= \Gamma_{1ii} = \Gamma_{i1i} = -\Gamma_{ii1} = \frac{1}{2} \vartheta' \text{ for } 1 < i, \\ \mathcal{R}^{g_1}(\partial_1, \partial_i) \partial_i &= \left\{ -\frac{1}{2} \vartheta^{-1} \vartheta'' + \frac{1}{2} \vartheta^{-2} \vartheta' \vartheta' \right\} \partial_1 \text{ for } 1 < i, \\ \mathcal{R}^{g_1}(\partial_i, \partial_j) \partial_j &= -\frac{1}{4} \vartheta^{-2} \vartheta' \vartheta' \partial_i \text{ for } 1 < i < j, \\ \text{Tr}_{g_1} \text{Ric}_{g_1} &= -(n-1) \vartheta^{-2} \vartheta'' + \vartheta^{-3} \left\{ (n-1) - \frac{1}{4} (n-1)(n-2) \right\} \vartheta' \vartheta' \\ &= (n-1) \left\{ -\vartheta^{-2} \Delta_g \vartheta - \frac{n-6}{4} \vartheta^{-3} \|\text{grad}_g \vartheta\|_g^2 \right\}, \\ n \kappa_{g_1} &= -\vartheta^{-2} \Delta_g \vartheta - \frac{n-6}{4} \vartheta^{-3} \|\text{grad}_g \vartheta\|_g^2. \end{aligned}$$

This completes the proof of Assertion (1) by showing:

$$a_1 = -1, \quad a_2 = -2, \quad a_3 = -\frac{n-6}{4}, \quad a_4 = -3.$$

Let (M, g, ∇) be an arbitrary Weyl manifold. We use a similar argument to prove Assertion (2). Again, a bit of thought shows there are universal constants so

$$\tau_g = n(n-1) \kappa_g + a_5 \nabla_k \phi^k + a_6 \|\phi\|_g^2 \quad (3.b)$$

where $a_i = a_i(n)$; Assertion (2) will follow by taking the special case where the metric is \tilde{g} . Again, we evaluate these constants using the method of universal examples. We take the reference background metric to be flat and $\phi = \vartheta(x_1) dx^1$ for some smooth function ϑ of one variable. We evaluate the universal coefficients in Equation (3.b) and complete the proof of Assertion (2) by calculating:

$$\begin{aligned} \nabla_x y &= \nabla_x^g y + \phi(x)y + \phi(y)x - g(x, y) \phi^\sharp, \\ \Gamma_{11}^1 &= \Gamma_{1i}^i = \Gamma_{i1}^1 = -\Gamma_{ii}^1 = \vartheta \text{ for } 1 < i, \\ \mathcal{R}(\partial_1, \partial_i) \partial_i &= -\vartheta' \partial_1 \text{ for } 1 < i, \\ \mathcal{R}(\partial_i, \partial_j) \partial_j &= -\vartheta^2 \partial_i \text{ for } 1 < i < j, \\ \tau_g &= -2(n-1) \vartheta' - (n-1)(n-2) \vartheta^2 \\ &= -2(n-1) \nabla_k^g \phi^k - (n-1)(n-2) \|\phi\|_g^2. \end{aligned}$$

This completes the proof by showing $a_5 = -2(n-1)$ and $a_6 = -(n-1)(n-2)$. \square

We defined the Weyl metric \tilde{g} by requiring that \tilde{g} is in the conformal class of g and so that $\tau_{\tilde{g}} = 1$. Conversely, of course, if g_1 is in the conformal class of g and if $\tau_{\tilde{g}} = c$ is constant, then g_1 is homothetic to \tilde{g} . We note that the associated Weyl-Schouten tensor is a gauge invariant where

$$\tilde{\sigma} = \frac{1}{n-2} \left[S \text{Ric} - \frac{1}{2(n-1)} \tilde{g} \right].$$

3.6. Global characterizations of the Weyl metric. Again, throughout this section, we assume $\tau_g > 0$ and normalize the metric so $\tau_{\tilde{g}} = 1$. We apply the relations given in Proposition 3.8 to obtain global characterizations of the Weyl metric within the conformal class in terms of the scalar curvatures $\kappa_{\tilde{g}}, \kappa_g$ and τ_g . We assume g is positive definite. We first examine the compact case:

Theorem 3.9. *Let N be a compact Riemannian manifold. The gauge invariant total scalar curvature of \tilde{g} gives an upper bound for the right hand side in terms of an arbitrary metric g within the conformal class, i.e.*

$$\int_N \kappa_{\tilde{g}} \omega_{\tilde{g}} \geq \int_N \kappa_g \cdot \tau_g^{\frac{n-2}{2}} \omega_g.$$

Equality holds if and only if $\tilde{g} = \tau_g \cdot g$ with $\tau_g = \text{const.}$, i.e. both metrics are homothetic.

Proof. The Riemannian volume forms are related by the identity: $\omega_{\tilde{g}} = \tau_g^{\frac{n}{2}} \cdot \omega_g$. This leads to the relation:

$$\begin{aligned} n \int_N \kappa_{\tilde{g}} \omega_{\tilde{g}} &= n \int_N \kappa_g \tau_g^{\frac{n-2}{2}} \omega_g - \int_N \tau_g^{\frac{n-4}{2}} \Delta_g \tau_g \omega_g \\ &\quad - \frac{1}{4}(n-6) \int_N \tau_g^{\frac{n-6}{2}} \cdot \|\text{grad}_g \tau_g\|_g^2 \omega_g. \end{aligned} \quad (3.c)$$

Let $\alpha := \frac{1}{2}(n-2)$. We then have:

$$\Delta_g \tau_g^\alpha = \alpha(\alpha-1) \tau_g^{\alpha-2} \|\text{grad}_g \tau_g\|_g^2 + \alpha \tau_g^{\alpha-1} \Delta_g \tau_g. \quad (3.d)$$

We use Equation (3.d) to see that:

$$0 = \int_N \tau_g^{\frac{1}{2}(n-4)} \Delta_g \tau_g \omega_g + \frac{1}{2}(n-4) \int_N \tau_g^{\frac{n-6}{2}} \|\text{grad}_g \tau_g\|_g^2 \omega_g. \quad (3.e)$$

Similarly Equation (3.c) and Equation (3.e) give:

$$n \int_N \kappa_{\tilde{g}} \omega_{\tilde{g}} = n \int_N \kappa_g \tau_g^{\frac{n-2}{2}} \omega_g + \frac{1}{4}(n-2) \int_N \tau_g^{\frac{1}{2}(n-6)} \|\text{grad}_g \tau_g\|_g^2 \omega_g.$$

As $\tau_g > 0$ by assumption, this implies the inequality; the discussion of equality then follows immediately. \square

Next, we study complete manifolds. In dimension $n \leq 6$, the foregoing relations allow to apply the well known maximum principle of Omori [9] and Yau [13] and also Yau's [14] extension of the harmonic map principle to complete manifolds.

Theorem 3.10. *Let $3 \leq n \leq 6$. Assume that the Riemannian manifold (N, g) is complete with non-negative Ricci tensor Ric_g . Assume that $\tau_g \in L^p$ for some $p > 1$.*

- (1) *If the scalar curvatures satisfy the inequality $\kappa_{\tilde{g}} - \tau_g^{-1} \kappa_g \leq 0$ then $\tilde{g} = \tau_g \cdot g$ with $\tau_g = \text{const.}$, i.e. both metrics are homothetic.*
- (2) *If there exists $0 \leq c \in \mathbb{R}$ with $\kappa_{\tilde{g}} - \tau_g^{-1} \kappa_g = c$, then $\tilde{g} = \tau_g \cdot g$ with $\tau_g = \text{const.}$, i.e. both metrics are homothetic.*

Proof. Suppose that $\kappa_{\tilde{g}} - \tau_g^{-1} \kappa_g \leq 0$. Since $(n-6) \leq 0$, it then follows from Proposition 3.8 that $\Delta_g \tau_g \geq 0$. Since τ_g is in L^p for some $p > 1$, the results of Yau cited above then show that τ_g is constant thus verifying Assertion (1).

To prove Assertion (2), we apply the Omori-Yau maximum principle to see that there exists a sequence of points of the manifold such that

$$\lim_k \tau_g(p_k) = \inf \tau_g, \quad \lim_k \|\text{grad}_g \tau_g\|_g^2(p_k) = 0, \quad \text{and} \quad \lim_k (\Delta_g \tau_g)(p_k) \geq 0.$$

This gives:

$$\begin{aligned} 0 \leq nc &= n \lim_k (\kappa_{\tilde{g}} - \tau_g^{-1} \kappa_g)(p_k) \\ &= -\tau_g^{-2} \Delta_g \tau_g + \frac{1}{4}(n-6) \tau_g^{-3} \|\text{grad}_g \tau_g\|_g^2 \leq 0, \end{aligned}$$

thus $c = 0$ and $\kappa_{\tilde{g}} = \tau_g^{-1} \kappa_g$. The PDE reduces to

$$\Delta_g \tau_g = -\tau_g^{-1} \frac{1}{4}(n-6) \cdot \|\text{grad}_g \tau_g\|_g^2 \geq 0.$$

Now we apply Yau as above. \square

3.7. Trivial Weyl manifolds. We have the following useful result that characterizes *trivial* Weyl manifolds:

Theorem 3.11. *Let $\mathcal{W} = (M, g, \nabla)$ be a Weyl manifold with $H^1(M; \mathbb{R}) = 0$. The following assertions are equivalent and if any is satisfied, we say that \mathcal{W} is trivial.*

- (1) $d\phi = 0$.
- (2) $\nabla = \nabla^{g_1}$ for some g_1 in the conformal class defined by g .
- (3) $\nabla = \nabla^{g_1}$ for some semi-Riemannian metric g_1 .
- (4) $R_P(\nabla) \in \mathfrak{A}$ for every $P \in M$.

Proof. Suppose that $d\phi = 0$. Since $H^1(M; \mathbb{R}) = 0$, we can express $\phi = df$ for some function f . Then ∇ is the Levi-Civita connection for the conformally equivalent metric $g_1 := e^{2f}g$. Thus Assertion (1) implies Assertion (2). Clearly Assertion (2) implies Assertion (3). Since the curvature tensor of the Levi-Civita connection is algebraic, Assertion (3) implies Assertion (4). Suppose that Assertion (4) holds. We apply Theorem 6 of [6] to see $d\phi = -\frac{1}{n}\Lambda \text{Ric}$. Since the curvature tensor is algebraic, $\Lambda \text{Ric} = 0$. Thus Assertion (4) implies Assertion (1). \square

In view of Theorem 3.11, Proposition 1.3 and Lemma 2.22 offer simple tools to verify, in terms of curvature decompositions, whether a Weyl manifold is trivial. The following Theorem characterizes trivial Weyl manifolds; for this we compare gauge invariant global scalar curvatures.

Theorem 3.12. *Let \mathcal{W} be a compact Weyl manifold without boundary, with positive definite Weyl metric \tilde{g} and associated Riemannian volume form $\omega_{\tilde{g}}$. Then the total volume of (N, \tilde{g}) satisfies*

$$\int_N \omega_{\tilde{g}} \leq n(n-1) \int_N \kappa_{\tilde{g}} \omega_{\tilde{g}}.$$

Equality holds if and only if ∇ is the Levi-Civita connection of \tilde{g} .

Proof. We integrate the second relation in Proposition 3.8 and apply Stokes' theorem. Equality gives $\tilde{\phi} = 0$ and thus $\nabla = \nabla^{\tilde{g}}$. \square

3.8. Einstein-Weyl manifolds. We recall the following well known definition.

Definition 3.13. A Weyl manifold (N, g, ∇) is said to be an *Einstein-Weyl manifold* if the Ricci tensor $\text{Ric} = \text{Ric}(\nabla)$ satisfies

$$S \text{Ric} = \lambda \cdot g. \quad (3.f)$$

Remark 3.14.

- (1) Obviously the foregoing relation is conformally invariant, and

$$n \cdot \lambda = \tau_g \text{ with } \tau_g = \text{Tr}_g \text{Ric} \text{ and } S \text{Ric} = \frac{1}{n} \tau_g \cdot g.$$

- (2) It follows from Theorem 1.1 that also $S \text{Ric}^* = \frac{1}{n} \tau_g \cdot g$ on Einstein-Weyl manifolds.

- (3) Let the Weyl metric \tilde{g} be well defined. Then \mathcal{W} is Einstein-Weyl if and only if

$$S \operatorname{Ric} = \frac{1}{n} \tau_{\tilde{g}} \cdot \tilde{g} = \frac{1}{n} \cdot \tilde{g}.$$

Thus, in particular, by renormalizing the metric, if $\tau > 0$, then we can assume that the Einstein multiple λ of Equation (3.f) satisfies $\lambda = +1$. Note that in the semi-Riemannian setting, one automatically has λ is constant if the dimension is at least 3.

We apply the decomposition results from Section 2.3 and immediately get the following characterizations of Einstein-Weyl manifolds in terms of components of the decompositions, where again $R = R(\nabla)$.

Proposition 3.15. *Let \mathcal{W} be a Weyl manifold. Then the following properties are equivalent:*

- | | |
|--|--|
| (1) \mathcal{W} is Einstein-Weyl. | (2) $R = (\alpha_1 + \alpha_6 + \alpha_4 + \alpha_5)(R)$. |
| (3) $R = (\pi_1 + \pi_3 + \pi_4 + \pi_6)(R)$. | (4) $\alpha_2(R) = 0$. |
| (5) $\pi_2(R) = 0$. | (6) $\pi_5(R) = 0$. |
| (7) $\operatorname{Ric}(\alpha_2(R)) = 0$. | (8) $\operatorname{Ric}^*(\alpha_2(R)) = 0$. |
| (9) $\operatorname{Ric}(\pi_2(R)) = 0$. | (10) $\operatorname{Ric}^*(\pi_2(R)) = 0$. |
| (11) $\operatorname{Ric}^*(\pi_5(R)) = 0$. | (12) $\operatorname{Ric}^*(\alpha_1(R)) = \operatorname{Ric}(R)$. |
| (13) $\operatorname{Ric}(\pi_1(R)) = \operatorname{Ric}(R)$. | (14) $\operatorname{Ric}^*(\pi_1(R)) = \operatorname{Ric}(R)$. |
| (15) $\operatorname{Ric}(\alpha_1(R)) = \operatorname{Ric}(R)$. | |

Remark 3.16.

- (1) The characterization of Einstein-Weyl manifolds in Assertion (5) generalizes a well known characterization of Einstein spaces within the class of semi-Riemann manifolds, using the decomposition of algebraic curvature tensors.
- (2) Trivially any Ricci-flat Weyl structure is Einstein-Weyl.

3.9. The conformal and the projective structure of Weyl manifolds. Introducing the concept of Weyl geometry, it was Weyl's intention to relate the conformal class with the projective structure of what we call the Weyl connection ∇ . Concerning the decompositions that we studied and the conformal class, we recall Section 3.2. For fixed metric g and concerning the projective structure, according to Section 2.6 the projective curvature tensor appears in the W -decomposition as

$$p(R) = \bigoplus_{i=4}^6 \pi_i(R);$$

to this decomposition there corresponds a gauge invariant decomposition of the curvature operator \mathcal{R} . Following Section 2.2, the Higa term can be expressed as

$$H(R) = \pi_3(R) + \pi_4(R).$$

These observations and Proposition 2.17 finally lead to the following Theorem which in particular generalizes a classical result, namely: *A projectively flat semi-Riemannian manifold has constant sectional curvature.*

Theorem 3.17. *Let (N, g, ∇) be a Weyl manifold with $H^1(M; \mathbb{R}) = 0$.*

- (1) *If $H(\mathcal{R}) = 0$ then the Weyl manifold is trivial.*
- (2) *If ∇ is projectively flat then the curvature tensor satisfies $R = \pi_1(R)$, i.e. R is of constant curvature type; in particular, \mathcal{W} is Einstein-Weyl. Moreover, $\Lambda \operatorname{Ric} = 0$, thus (N, g, ∇) is trivial.*

4. Length curvature and directional curvature

Let $\mathcal{W} = (N, g, \nabla)$ be a Weyl-manifold. We use Theorem 1.1 to see:

$$R(x, y, z, w) + R(x, y, w, z) = 2F(x, y)g(z, w),$$

where the operator F satisfies $F(x, y) := -\frac{2}{n}(\Lambda \text{Ric})(x, y) = -2d\phi(x, y)$; here d denotes exterior derivation. Parts of the following can be found in Section 1 of [8].

Definition 4.1. F is called the *length curvature*, and

$$\mathcal{K}(x, y)z := \mathcal{R}(x, y)z - F(x, y)z$$

is called the *directional curvature* of \mathcal{W} . Perlick [10] explains this terminology.

Lemma 4.2. F and \mathcal{K} are gauge invariant.

For $R = R(\nabla) \in \mathfrak{W}$, according to Section 1.4, we consider its conjugate curvature tensor R^* . We write $g(\mathcal{K}(x, y)z, w) =: K(x, y, z, w)$. Recall that the length curvature operator \mathcal{F} was defined by setting $\mathcal{F}(x, y)z := F(x, y)z$. Then the foregoing definitions immediately give:

Lemma 4.3.

- (1) $R^*(x, y, w, z) = R(x, y, w, z) - 2F(x, y)g(z, w)$.
- (2) \mathcal{R}^* is gauge invariant.
- (3) $\mathcal{F} = \frac{1}{2}(\mathcal{R} - \mathcal{R}^*)$.
- (4) $\mathcal{K} = \frac{1}{2}(\mathcal{R} + \mathcal{R}^*)$.
- (5) $g(\mathcal{K}(x, y)z, w) + g(\mathcal{K}(x, y)w, z) = 0$.

Corollary 4.4. Adopt the notation given above. We have the relations:

$$\begin{aligned} g(\mathcal{F}(x, y)z, w) &= \frac{1}{2}(R - R^*)(x, y, z, w) = \frac{1}{2}(H(A)(x, y, z, w) + H(A)(x, y, w, z)), \\ g(\mathcal{K}(x, y)z, w) &= \frac{1}{2}(R + R^*)(x, y, z, w) = \\ &= (\alpha_1 + \alpha_2 + \alpha_6)(R)(x, y, z, w) + \frac{1}{2}(H(A)(x, y, z, w) - H(A)(x, y, w, z)). \end{aligned}$$

The relations in (3) and (4) in the foregoing Lemma show a “symmetry” in the definition of \mathcal{K} and \mathcal{F} and thus justify consideration of the conjugate curvature operator \mathcal{R}^* (conjugate curvature tensor R^* , respectively) in Weyl geometry. Moreover, the foregoing Corollary clarifies the role of the Higa term for \mathcal{K} and \mathcal{F} . From Proposition 1.3 recall that, in general, \mathcal{K} does not satisfy the Bianchi identity, it is not algebraic. The following Lemma summarizes simple characterizations of trivial Weyl manifolds.

Lemma 4.5. We have the equivalences:

- (1) $F = 0$. (2) $H = 0$. (3) $R = R^*$.
- (4) $R = K$. (5) $\Lambda \text{Ric} = 0$. (6) $\alpha_4(R) = 0$.
- (7) $\alpha_5(R) = 0$. (8) $\pi_3(R) = 0$. (9) $\pi_4(R) = 0$.
- (10) \mathcal{R}^* satisfies (1.c). (11) \mathcal{F} satisfies (1.c). (12) \mathcal{K} satisfies (1.c).
- (13) $R \in \mathfrak{A}$. (14) $R^* \in \mathfrak{A}$.

- (15) There exists a metric g_1 in the conformal class of \mathcal{W} such that its Levi-Civita connection ∇^{g_1} coincides with the Weyl connection.

With the results from Section 2 in [5] we get:

Corollary 4.6. The Ricci tensors $\text{Ric} = \text{Ric}(R)$ and $\text{Ric}^* = \text{Ric}^*(R)$ satisfy:

- (1) $\text{Ric}(\mathcal{K}) = \frac{1}{2}(\text{Ric} + \text{Ric}^*)$. (2) $S \text{Ric}(\mathcal{K}) = S \text{Ric}$.
- (3) $\Lambda \text{Ric}(\mathcal{K}) = \frac{1}{n}(n - 2)\Lambda \text{Ric}$. (4) $\text{Ric}(\mathcal{F}) = \frac{1}{n}\Lambda \text{Ric}$.
- (5) $S \text{Ric}(\mathcal{F}) = 0$.

Lemma 4.7. *Let ∇ be projectively flat. Then $F = 0$.*

Proof. Projective flatness implies $\Lambda \text{Ric} = 0$, thus $F = 0$. \square

Proposition 4.8. *Let \mathcal{W} be an n -dimensional Weyl manifold such that the length curvature is non-zero at least at one point. The following assertions are equivalent:*

- (1) $n = 4$. (2) $\Lambda \text{Ric}^* = 0$.

Remark 4.9. Let $n \neq 4$. Then $\Lambda(\text{Ric}) = 0$ if and only if $\Lambda(\text{Ric}^*) = 0$.

ACKNOWLEDGMENTS

Research of P. Gilkey partially supported by DFG PI 158/4-6 (Germany) and by project MTM2009-07756 (Spain). Research of S. Nikčević partially supported by a research grant of the TU Berlin, by project MTM2009-07756 (Spain), and by 144032 (Serbia). Research of U. Simon partially supported by DFG PI 158/4-6 (Germany). We thank V. Perlick for hints.

REFERENCES

- [1] N. Blažić, P. Gilkey, S. Nikčević, and U. Simon, Algebraic theory of affine curvature tensors, *Archivum Mathematicum*, Masaryk University (Brno, Czech Republic) ISSN 0044-8753, tomas **42** (2006), supplement: Proceedings of the 26th Winter School Geometry and Physics 2006 (SRNI), 147–168.
- [2] N. Boman, On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection, *Rendiconti del Circolo Matematico di Palermo* **XXIX** (1990), 331–380.
- [3] E. Canfes, On generalized recurrent Weyl spaces and Wong’s conjecture, *Differ. Geom. Dyn. Syst.* **8**, 34–42.
- [4] L. P. Eisenhart, Non-Riemannian Geometry, AMS Colloquium Publications **VIII**, 5-th printing, N.Y. 1964.
- [5] P. Gilkey, S. Nikčević, and U. Simon, Geometric theory of equiaffine curvature tensors, *Results Math.* **56** (2009), 275–318.
- [6] P. Gilkey, S. Nikčević, and U. Simon, Geometric realizations, curvature decompositions, and Weyl manifolds, <http://arXiv.org/abs/1002.5027>.
- [7] T. Higa, Weyl manifolds and Einstein-Weyl manifolds, *Comm. Math. Univ. St. Pauli* **42** (1993), 143–160.
- [8] T. Higa, Curvature tensors and curvature conditions in Weyl geometry, *Comm. Math. Univ. St. Pauli* **43** (1994), 139–153.
- [9] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan* **19** (1967), 205–241.
- [10] V. Perlick, Observer fields in Weylian space time models, *Class. Quantum Grav.* **8** (1991), 1369–1385.
- [11] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, *1969 Global Analysis (Papers in honor of K. Kodaira)*, Univ. Tokyo Press, Tokyo, 355–365.
- [12] H. Weyl, *Space-Time-Matter*, Dover Publ. 1922.
- [13] S. T. Yau, Harmonic functions on complete Riemannian manifolds *Commun. Pure Appl. Math.*, **28** (1975), 201–228.
- [14] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, *Indiana Univ. Math. J.* **25** (1976), 659–670.

PG: MATHEMATICS DEPARTMENT, UNIVERSITY OF OREGON, EUGENE OR 97403 USA,
E-MAIL: GILKEY@UOREGON.EDU

SN: MATHEMATICAL INSTITUTE, SANU, KNEZ MIHAILOVA 36, P.P. 367, 11001 BELGRADE,
SERBIA., EMAIL: STANAN@MI.SANU.AC.RS

US: INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17. JUNI 135, D-10623 BERLIN, GERMANY, EMAIL: SIMON@MATH.TU-BERLIN.DE